Indian Statistical Institute
Backpaper Exam
Algebra-I

Time : 3 hours Max. Marks : 100

Answer all questions.

(1) Give examples of
(a) An infinite group in which every element has finite order and for each positive integer n there is an element of order n.
(b) A group G such that every finite group is isomorphic to some subgroup of G.
(c) A group G such that $G \cong G \times G$.
(d) A non-abelian group G such that all subgroups of G are normal in G.
(e) A group G such that $\mathcal{A}ut\ G \cong G$.
(10)

(2) Let G be a group. Show that if $G/Z(G)$ is cyclic, then G is abelian.
(15)

(3) Let G be a group and A, B be subgroups of G such that $A \subseteq N_G(B)$. Then show that $A \cap B$ is a normal subgroup of A and $AB/B \cong A/A \cap B$.
(15)

(4) Show that S_n is generated by the n cycle $(1\ 2 \dots n)$ and the transposition $(1\ 2)$. Is it true for any transposition and any n cycle?
(15)

(5) Let G be a group of order $2m$ where m is odd. Show that G contains a normal subgroup of order m.
(15)

(6) Show that a group of order 12 either contains a normal subgroup of order 3 or is isomorphic to A_4.
(15)

(7) Show that A_n is the only subgroup of index 2 in S_n for all $n \geq 3$.
(15)